Features of (Un)decidable Logics

Søren Brinck Knudstorp
ILLC, University of Amsterdam

February 28, 2025

Student Logic Colloquium

Plan for the talk

- (Un)decidability: what and why?
- Propositional team logics and their decidability
- Exploring boundaries between the decidable and the undecidable
    - Solving problems and obtaining insights along the way
    - Using insights to solve one last problem

(Un)decidability: what and why?

What?

A decision problem is a collection of inputs I, with a yes-or-no question for each $i \in I$.

A decision problem is decidable if there is an algorithm that, given any $i \in I$, accurately answers the question. Otherwise, it is undecidable.

A logic $\mathbf{L}$, in a language $\mathcal{L}$, is decidable if there is an algorithm that, given any $\varphi \in \mathcal{L}$, determines whether $\varphi \in \mathbf{L}$. Otherwise, it is undecidable.

Why?

(Un)decidability: what and why?

What?

A decision problem is a collection of inputs I, with a yes-or-no question for each $i \in I$.

A decision problem is decidable if there is an algorithm that, given any $i \in I$, accurately answers the question. Otherwise, it is undecidable.

A logic $\mathbf{L}$, in a language $\mathcal{L}$, is decidable if there is an algorithm that, given any $\varphi \in \mathcal{L}$, determines whether $\varphi \in \mathbf{L}$. Otherwise, it is undecidable.

Why? *Because it is a profound conceptual distinction.*

Propositional team logics and their decidability

Traditionally (in, e.g., CPL), formulas φ are evaluated at single valuations $v : \mathbf{Prop} \to \{0, 1\}$,

$$v \vDash \varphi.$$

In team semantics, formulas φ are evaluated at sets ('teams') of valuations $s \subseteq \{v \mid v : \mathbf{Prop} \to \{0, 1\}\}$,

$$s \vDash \varphi.$$

Definition (some team-semantic clauses)

Let $X := \{v \mid v : \mathbf{Prop} \to \{0, 1\}\}$. For $s \in \mathcal{P}(X)$, we define

$s \vDash p$	iff	$\forall v \in s : v(p) = 1$,
$s \vDash \varphi \wedge \psi$	iff	$s \vDash \varphi$ *and* $s \vDash \psi$,
$s \vDash \varphi \mathbin{\rotatebox[origin=c]{180}{$\mathbb{A}$}} \psi$	iff	$s \vDash \varphi$ *or* $s \vDash \psi$,
$s \vDash {\sim}\varphi$	iff	$s \nvDash \varphi$,
$s \vDash \varphi \vee \psi$	iff	there exist $s', s'' \in \mathcal{P}(X)$ such that $s' \vDash \varphi$; $s'' \vDash \psi$; and $s = s' \cup s''$.

Observation. All propositional team logics are decidable: given φ, simply check whether $s \vDash \varphi$ for all $s \subseteq \{v \mid v : \mathbf{Prop}(\varphi) \to \{0, 1\}\}$.

Yet, this explanation is hardly satisfactory.
What is it that makes propositional team logics decidable, *and others not?*

Team semantics as relational semantics

Recall our semantic clauses: For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$s \vDash p$	iff	$\forall v \in s : v(p) = 1$,
$s \vDash \varphi \wedge \psi$	iff	$s \vDash \varphi$ *and* $s \vDash \psi$,
$s \vDash \varphi \mathbin{\mathbb{V}} \psi$	iff	$s \vDash \varphi$ *or* $s \vDash \psi$,
$s \vDash {\sim}\varphi$	iff	$s \nvDash \varphi$,
$s \vDash \varphi \vee \psi$	iff	there exist $s', s'' \in \mathcal{P}(X)$ such that $s' \vDash \varphi$; $s'' \vDash \psi$; and $s = s' \cup s''$.

Team semantics as relational semantics

Recall our semantic clauses: For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$$
\begin{array}{lll}
s \vDash p & \text{iff} & \forall v \in s : v(p) = 1, \\
s \vDash \varphi \wedge \psi & \text{iff} & s \vDash \varphi \ \textit{ and } \ s \vDash \psi, \\
s \vDash \varphi \mathbin{\color{orange}{\bigvee}} \psi & \text{iff} & s \vDash \varphi \ \textit{ or } \ s \vDash \psi, \\
s \vDash {\sim}\varphi & \text{iff} & s \nvDash \varphi, \\
s \vDash \varphi \vee \psi & \text{iff} & \text{there exist } s', s'' \in \mathcal{P}(X) \text{ such that } s' \vDash \varphi; \\
 & & s'' \vDash \psi; \text{ and } s = s' \cup s''.
\end{array}
$$

Team semantics as relational semantics

Recall our semantic clauses: For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$$
\begin{array}{lll}
s \vDash p & \text{iff} & \forall v \in s : v(p) = 1, \\
s \vDash \varphi \land \psi & \text{iff} & s \vDash \varphi \ \textit{and} \ s \vDash \psi, \\
s \vDash \varphi \vee \psi & \text{iff} & s \vDash \varphi \ \textit{or} \ s \vDash \psi, \\
s \vDash \neg\varphi & \text{iff} & s \nvDash \varphi, \\
s \vDash \varphi \vee \psi & \text{iff} & \text{there exist } s', s'' \in \mathcal{P}(X) \text{ such that } s' \vDash \varphi; \\
 & & s'' \vDash \psi; \text{ and } s = s' \cup s''.
\end{array}
$$

Team semantics as relational semantics

Recall our semantic clauses: For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$$\begin{array}{lll} s \vDash p & \text{iff} & \forall v \in s : v(p) = 1, \\ s \vDash \varphi \wedge \psi & \text{iff} & s \vDash \varphi \;\; \textit{and} \;\; s \vDash \psi, \\ s \vDash \varphi \vee \psi & \text{iff} & s \vDash \varphi \;\; \textit{or} \;\; s \vDash \psi, \\ s \vDash \neg\varphi & \text{iff} & s \nvDash \varphi, \\ s \vDash \varphi \circ \psi & \text{iff} & \text{there exist } s', s'' \in \mathcal{P}(X) \text{ such that } s' \vDash \varphi; \\ & & s'' \vDash \psi; \text{ and } s = s' \cup s''. \end{array}$$

This induces a powerset frame $\mathbb{F} = (\mathcal{P}(X), \cup)$, where '$\circ$' is a binary modality referring to the ternary $\cup$-relation: $s = s' \cup s''$;

Team semantics as relational semantics

Recall our semantic clauses: For $X := \{v \mid v : \mathbf{Prop} \to \{0,1\}\}$ and $s \in \mathcal{P}(X)$, we had

$$
\begin{array}{lll}
s \vDash p & \text{iff} & \forall v \in s : v(p) = 1, \\
s \vDash \varphi \wedge \psi & \text{iff} & s \vDash \varphi \ \textit{and} \ s \vDash \psi, \\
s \vDash \varphi \vee \psi & \text{iff} & s \vDash \varphi \ \textit{or} \ s \vDash \psi, \\
s \vDash \neg\varphi & \text{iff} & s \nvDash \varphi, \\
s \vDash \varphi \circ \psi & \text{iff} & \text{there exist } s', s'' \in \mathcal{P}(X) \text{ such that } s' \vDash \varphi; \\
 & & s'' \vDash \psi; \text{ and } s = s' \cup s''.
\end{array}
$$

This induces a powerset frame $\mathbb{F} = (\mathcal{P}(X), \cup)$, where '$\circ$' is a binary modality referring to the ternary $\cup$-relation: $s = s' \cup s''$; and a model $\mathbb{M} = (\mathcal{P}(X), \cup, V)$ with a 'principal valuation', i.e.,

$$V(p) := \{s \in \mathcal{P}(X) \mid \forall v \in s : v(p) = 1\} = {\downarrow}\{v \in X \mid v(p) = 1\}.$$

In fact, if we take all powerset frames $(\mathcal{P}(X), \cup)$, redefine the base clause

$$(\mathcal{P}(X), \cup, V), s \Vdash p \qquad \text{iff} \qquad s \in V(p),$$

and only allow principal valuations $V : \mathbf{Prop} \to \{{\downarrow}s \mid s \in \mathcal{P}(X)\}$, we get sound and complete relational semantics for team logics.

Proof. A simple p-morphism argument.

Powerset frames and Boolean frames

Summarizing, (i) team logics are decidable, and (ii) relational semantics for team logics are given by powerset frames $(\mathcal{P}(X), \cup)$ with principal valuations $V : \mathbf{Prop} \to \{\downarrow s \mid s \in \mathcal{P}(X)\}$.

Question: *Sticking with the signature $\{\wedge, \vee, \neg, \circ\}$, what happens if we allow for arbitrary valuations $V : \mathbf{Prop} \to \mathcal{PP}(X)$? Does the logic remain decidable?*

In fact, this question is intimately related with an open problem: Goranko and Vakarelov (1999) consider the logic of Boolean frames – instead of a powerset $\mathcal{P}(X)$, the carrier is a Boolean algebra B – and raises the problem of its decidability.[1]

Theorem

The logic of powerset frames, in the signature $\{\wedge, \vee, \neg, \circ\}$, with arbitrary valuations is **undecidable**. And so is the hyperboolean modal logic of Goranko and Vakarelov (1999).

[1] Goranko and Vakarelov (1999) call their logic 'hyperboolean modal logic' and include modalities for all the Boolean operations, not just the join.

Proof method: tiling

- A (Wang) tile is a square with colors on each side.
- The tiling problem: given any finite set of tiles $\mathcal{W}$, determine whether each point in the quadrant $\mathbb{N}^2$ can be assigned a tile from $\mathcal{W}$ such that neighboring tiles share matching colors on connecting sides.
- The tiling problem was introduced by Wang (1963) and proven undecidable by Berger (1966).

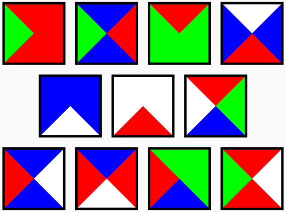


Figure 1: Wang tiles

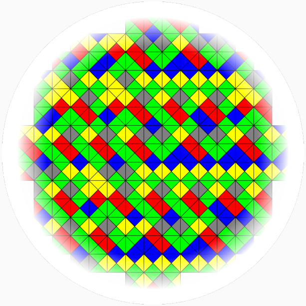


Figure 2: A tiling of the plane

Figures taken from: https://en.wikipedia.org/wiki/Wang_tile

Proof method: tiling

Theorem

The logic of powerset frames, in the signature $\{\wedge, \vee, \neg, \circ\}$, with arbitrary valuations is **undecidable**. And so is the hyperboolean modal logic of Goranko and Vakarelov (1999).

Proof idea.

For each finite set of tiles $\mathcal{W}$, we construct a formula $\phi_{\mathcal{W}}$ such that $\mathcal{W}$ tiles the quadrant if and only if $\phi_{\mathcal{W}}$ is satisfiable. $\square$

Dividing the proof into two lemmas, corresponding to a direction each, we can prove both results in one go:

Lemma

If $\phi_{\mathcal{W}}$ is satisfiable (in a Boolean frame), then $\mathcal{W}$ tiles $\mathbb{N}^2$.

Lemma

If $\mathcal{W}$ tiles $\mathbb{N}^2$, then $\phi_{\mathcal{W}}$ is satisfiable (in $(\mathcal{P}(\mathbb{N}), \cup)$).

Insight 1: valuations matter

Semilattice frames, associativity and negation

Question: *Since we can weaken from powersets to Boolean algebras and stay undecidable, how much further can we go while remaining undecidable?*

Weakening from powersets $(\mathcal{P}(X), \cup)$ to general (join-)semilattices $(S, \sqcup)$, we get a problem posed by Bergman (2018) and Jipsen et al. (2021) (and SBK (2023a)).

Theorem

For any class of semilattices containing $(\mathcal{P}(\mathbb{N}), \cup)$, its logic in the signature $\{\wedge, \vee, \neg, \circ\}$, is undecidable.

Proof. *Formulas in handout (manuscript with proof available on request)*

Semilattice frames, associativity and negation

Question: *Since we can weaken from powersets to Boolean algebras and stay undecidable, how much further can we go while remaining undecidable?*

Weakening from powersets $(\mathcal{P}(X), \cup)$ to general (join-)semilattices $(S, \sqcup)$, we get a problem posed by Bergman (2018) and Jipsen et al. (2021) (and SBK (2023a)).

Theorem

For any class of semilattices containing $(\mathcal{P}(\mathbb{N}), \cup)$, its logic in the signature $\{\wedge, \vee, \neg, \circ\}$, is undecidable.

Question: *What if we weaken even further than semilattices?*

Semilattice frames, associativity and negation

Question: *Since we can weaken from powersets to Boolean algebras and stay undecidable, how much further can we go while remaining undecidable?*
Weakening from powersets $(\mathcal{P}(X), \cup)$ to general (join-)semilattices $(S, \sqcup)$, we get a problem posed by Bergman (2018) and Jipsen et al. (2021) (and SBK (2023a)).

Theorem

For any class of semilatticesassociative framesassociative frames containing $(\mathcal{P}(\mathbb{N}), \cup)$, its logic in the signature $\{\wedge, \vee, \neg, \circ\}$, is undecidable.

Question: *What if we weaken even further than semilattices?*
(Partial) answer 1: As semilattices also are partial orders '$\leq$' with all binary suprema, we could consider the logic of all *partial orders simpliciter*. This is modal information logic, which is proven decidable in SBK (2023b).

Answer 2: As semilattices are associative, commutative, idempotent functions, we could also consider the logic of all associative ternary relations. This is undecidable (Kurucz et al. 1995).

Question: *What if we, instead, reduce our signature* $\{\wedge, \vee, \neg, \circ\}$*?*
Answer: If we stick to semilattices but *omit negation*, so signature is $\{\wedge, \vee, \circ\}$, we obtain *Finean truthmaker semantics*, proven decidable in SBK (2023a).

Insight 2: associativity matters

Insight 3: negation matters

(Un)decidability of relevant $\mathbf{S}$: using our insights

Problem of concern: *Is relevant logic* $\mathbf{S}$ *decidable?*

$\mathbf{S}$ is the logic of semilattice frames $(S, \sqcup, \mathbf{0})$ with a bottom element $\mathbf{0}$, with arbitrary valuations, in the signature $\{\wedge, \vee, \rightarrow\}$. '$\rightarrow$' is closely connected to '$\circ$' (it is its residual).

What we know about the problem:

- Omitting disjunction, the logic $\mathbf{S}_{\wedge,\rightarrow}$ is decidable.
- If we restrict to persistent valuations, we obtain negation-free intuitionistic logic, which is decidable.
- $\mathbf{S}$ is closely connected to the relevant logic $\mathbf{R}^{+}$, which is undecidable.
    - Und. of $\mathbf{R}^{+}$ was shown by Urquhart (1984), but $\mathbf{S}$ eluded these techniques.
    - Eventually, this led experts, including Urquhart (2016), to conjecture that $\mathbf{S}$ is decidable.

What we notice about the problem:

- *Valuations are arbitrary*, contra negation-free intuitionistic logic. ['suggesting' undecidability]
- $\mathbf{S}$ is *negation-free*! [suggesting decidability]
- Frames of $\mathbf{S}$ are semilattices, *they are associative*! [suggesting undecidability]

Theorem: $\mathbb{S}$ is undecidable

Theorem: $\mathbb{S}$ is undecidable[2]

[2] See SBK (2024)

Relevant S is undecidable: Proof idea

Theorem: S is undecidable.

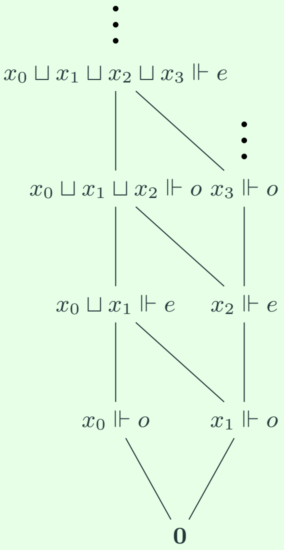
We cover the no-FMP proof instead, since it is considerably simpler than the undecidability proof, yet effectively illustrates some of the same key ideas.[3]

Theorem: S lacks the FMP.

Proof. We show that the formula ψ_∞ from the handout only is refuted by infinite models.

[3] Additionally, it addresses an open problem (as recently raised in Weiss 2021)

Refuting model



What about R?

When Urquhart (1984) proved $\mathbf{R}$ (and $\mathbf{E}$ and $\mathbf{T}$) undecidable, he concluded by remarking "The undecidability results [...] omit one notable case. This is the logic consisting of all formulas valid in the semilattice semantics [...] The decision problem for this system is still open."

What about $\mathbf{R}$?

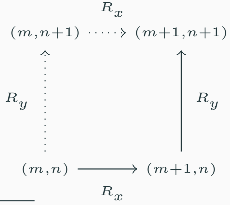
When Urquhart (1984) proved $\mathbf{R}$ (and $\mathbf{E}$ and $\mathbf{T}$) undecidable, he concluded by remarking "The undecidability results [...] omit one notable case. This is the logic consisting of all formulas valid in the semilattice semantics [...] The decision problem for this system is still open."

Question: *While $\mathbf{S}$ escaped those techniques, can we extend the present proof to include $\mathbf{R}$ (and $\mathbf{E}$ and $\mathbf{T}$) as well?*

Answer: Yes! It extends to $\mathbf{R}^+$, hence $\mathbf{R}$, as well ($\mathbf{E}$ and $\mathbf{T}$ to be checked).

Two comments on the proof:

1. We use the $n \times n$, for all $n \in \mathbb{N}$, tiling problem instead.
2. On associativity and tiling (modulo commutativity):
   - Associativity for ternary relations: $R(ab)cd \Rightarrow Ra(bc)d$.[4]
   - Write $aR_b c$ for $Rabc$. Then $R(ab)cd$ means $\exists e{:}aR_b eR_c d$; and $R(ac)bd$ means $\exists f{:}aR_c fR_b d$.
   - So sp. $(m,n)R_x(m+1,n)R_y(m+1,n+1)$. From associativity, we get that there is a point $(m,n+1)$ s.t. $(m,n)R_y(m,n+1)R_x(m+1,n+1)$. I.e.:



[4] Mod comm., it is $R(ab)cd \Rightarrow R(ac)bd$. $R(ab)cd$ means $\exists e{:}Rabe \wedge Recd$; and $R(ac)bd$ means $\exists f{:}Racf \wedge Rfbd$.

Question: Is Bunched Implication Logic (BI) decidable?

Theorem: BI is undecidable

Conclusion

We obtained new (undecidability) results, including:

- Hyperboolean modal logic is undecidable.[5]
- Modal logic of semilattices is undecidable.[6]
- $\mathbf{S}$ is undecidable [cf. SBK 2024].[7]
- (and a new proof of Urquhart (1984)'s result that $\mathbf{R}$ is undecidable)
- BI is undecidable.[8]

We compared them with known decidability results:

- Propositional team logics are decidable.
- Modal information logic is decidable [cf. SBK 2023b].[9]
- Truthmaker logics are decidable [cf. SBK 2023a].

Core messages:

- Valuations matter.
- Associativity matters.
- Negation matters, but we only needed a tiny bit of meta-language 'it is not the case that'.

---

[5] Raised in Goranko and Vakarelov (1999)

[6] Raised in Bergman (2018), Jipsen et al. (2021), and SBK (2023b)

[7] Raised by Urquhart (1972, 1984)

[8] Claimed decidable multiple times, first in Galmiche et al. (2005)

[9] Raised in van Benthem (2017, 2019)

References I

Berger, R. (1966). The undecidability of the domino problem. English. Vol. 66. Mem. Am. Math. Soc. Providence, RI: American Mathematical Society (AMS). DOI: 10.1090/memo/0066 (cit. on p. 13).

Bergman, C. (2018). "Introducing Boolean Semilattices". In: Don Pigozzi on Abstract Algebraic Logic, Universal Algebra, and Computer Scie Ed. by J. Czelakowski. Springer, pp. 103–130 (cit. on pp. 16 sqq., 30).

Galmiche, D., D. Mery, and D. Pym (2005). "The semantics of BI and resource tableaux". In: Mathematical Structures in Computer Science 15.6, pp. 1033–1088. DOI: 10.1017/S0960129505004858 (cit. on p. 30).

Goranko, V. and D. Vakarelov (1999). "Hyperboolean Algebras and Hyperboolean Modal Logic". In: Journal of Applied Non-Classical Logics 9.2-3, pp. 345–368. DOI: 10.1080/11663081.1999.10510971 (cit. on pp. 12, 14, 30).

References II

Jipsen, P., M. Eyad Kurd-Misto, and J. Wimberley (2021). **"On the Representation of Boolean Magmas and Boolean Semilattices".** In: Hajnal Andréka and István Németi on Unity of Science: From Computing to Rel Ed. by J. Madarász and G. Székely. Cham: Springer International Publishing, pp. 289–312. DOI: 10.1007/978-3-030-64187-0_12 (cit. on pp. 16 sqq., 30).

Knudstorp, S. B. (2024). **"Relevant S is Undecidable".** In: Proceedings of the 39th Annual ACM/IEEE Symposium on Logic in Computer Sc LICS '24. Tallinn, Estonia: Association for Computing Machinery. DOI: 10.1145/3661814.3662128. URL: https://doi.org/10.1145/3661814.3662128 (cit. on pp. 23, 30).

Knudstorp, S. B. (2023a). **"Logics of Truthmaker Semantics: Comparison, Compactness and Decidability".** In: Synthese (cit. on pp. 16 sqq., 30).

References III

- Knudstorp, S. B. (2023b). **"Modal Information Logics: Axiomatizations and Decidability".** In: Journal of Philosophical Logic (cit. on pp. 18, 30).
- Kurucz, Á. et al. (1995). **"Decidable and undecidable logics with a binary modality".** In: Journal of Logic, Language and Information 4, pp. 191–206 (cit. on p. 18).
- Urquhart, A. (1972). **"Semantics for relevant logics".** In: Journal of Symbolic Logic 37, pp. 159 –169 (cit. on p. 30).
- — (1984). **"The undecidability of entailment and relevant implication".** In: Journal of Symbolic Logic 49, pp. 1059 –1073 (cit. on pp. 21, 25 sq., 30).
- — (2016). **"Relevance Logic: Problems Open and Closed".** In: The Australasian Journal of Logic 13 (cit. on p. 21).

- Van Benthem, J. (10/2017). **“Constructive agents”.** In: Indagationes Mathematicae 29. DOI: 10.1016/j.indag.2017.10.004 (cit. on p. 30).
- — (2019). **“Implicit and Explicit Stances in Logic”.** In: Journal of Philosophical Logic 48.3, pp. 571–601. DOI: 10.1007/s10992-018-9485-y (cit. on p. 30).
- Wang, H. (1963). **“Dominoes and the ∀∃∀ case of the decision problem”.** In: Mathematical Theory of Automata, pp. 23–55 (cit. on p. 13).
- Weiss, Y. (2021). **“A Conservative Negation Extension of Positive Semilattice Logic Without the Finite Model Property”.** In: Studia Logica 109, pp. 125–136 (cit. on p. 24).

Thank you!

How can we think of this algebraically?

From relations to algebras

Given any set A with a ternary relation R, we can form the complex algebra:

$$(\mathcal{P}A, \cap, \cup, {}^c, \circ),$$

where

$$B \circ C := \{a \in A \mid Rabc, b \in B, c \in C\}.$$

The result is a Boolean algebra with an operator $\circ$.

In our case, A is the powerset $\mathcal{P}X$ and R the union relation $\cup$, so we get

$$(\mathcal{P}\mathcal{P}X, \cap, \cup, {}^c, \circ),$$

where

$$B \circ C := \{a \in A \mid a = b \cup c, b \in B, c \in C\}.$$

Finally,

- Let Pow^+ denote the class of complex algebras of powersets with union.
- Team logic is the theory of Pow^+ where homomorphisms send variables to principal downsets.
- If *arbitrary* homomorphisms, then we get $\mathbf{V}(\mathit{Pow}^+)$.